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Deviations from critical density in the generalised hard hexagon model

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Abstract. Ramanujan's elliptic function identities and others are used to obtain infinite product forms for the deviations $\rho-\rho_c$ from the critical density $\rho_c=\frac{1}{2}(1-5^{-1/2})$ in the four regimes of the generalised hard hexagon model. Product forms are also obtained for the singular part ρ_{sing} of the density confirming the simple scaling forms recently conjectured by Huse.

1. Introduction

The generalised hard hexagon model is actually a special class of hard square lattice gas models with diagonal interactions L and M obtained by restricting the activity z to the values

$$z=(1-e^{-L})(1-e^{-M})/(e^{L+M}-e^{L}-e^{M})\geqslant 0. \tag{1.1}$$

This constraint defines a surface, in the three-dimensional thermodynamic (z, L, M) space, on which the hard square models can be solved exactly (Baxter 1980, 1981, 1982, Baxter and Pearce 1982, 1983).

The two-dimensional surface (1.1) in fact consists of three disjoint sheets and the exact solution reveals a line of critical points on each sheet. On the first sheet ($L\geqslant 0$, $M\leqslant 0$), the line of critical points is located by the additional equation

$$z^{-1/2}(1-z\,e^{L+M})=[\tfrac{1}{2}(1+\sqrt{5})]^{-5/2} \tag{1.2a}$$

and separates a disordered fluid phase (regime I) from a triangular (3×1) ordered solid phase (regime II). This line, which includes the critical point ($L\to 0$, $M\to-\infty$, $z=\frac{1}{2}(11+5\sqrt{5})$) of the pure hard hexagon model ($L\to 0$, $M\to-\infty$, $z\geqslant 0$ arbitrary), appears to be a line of three-state Potts-like critical points. The line of critical points on the second sheet ($L\geqslant 0$, $M\geqslant 0$) is located by the equation

$$z^{-1/2}(1-z\,e^{L+M})=-[\tfrac{1}{2}(1+\sqrt{5})]^{-5/2} \tag{1.2b}$$

and separates a (first-order coexistence) surface of triple points (regime III) from a square ($\sqrt{2}\times\sqrt{2}$) ordered solid phase (regime IV). This line of critical points is actually a line of tricritical points at which the melting of the $\sqrt{2}\times\sqrt{2}$ solid phase changes over from continuous to first-order (Huse 1982, Baxter and Pearce 1983). The third sheet ($L\leqslant 0$, $M\geqslant 0$) differs from the first sheet only in the interchange of L and M corresponding to rotating the lattice by $\pi/2$.

On the basis of the exact solution on the *two*-dimensional manifold (1.1), Huse (1983) has recently examined the scaling behaviour of the hard square lattice gas with diagonal interactions in the larger context of the full *three*-dimensional (z, L, M) space. This led Huse (1983) to propose simple scaling forms for the free energy, spontaneous order parameters and density functions. Unfortunately, unlike the other exact results, the expressions previously given for the density functions in regimes I–IV are not in a form suitable for analysing critical scaling behaviour. In this paper we rectify this situation. More specifically, in each of the four regimes, we obtain convenient infinite product forms for the deviations of the density ρ from its critical value. In addition, we use these to obtain the singular part ρ_{sing} of the density in product form, thereby confirming the scaling forms conjectured by Huse (1983).

2. Definitions and results

By now the interplay (Baxter 1981, Andrews 1981, Andrews *et al* 1984) between the various density functions of hard hexagon type models and Rogers–Ramanujan identities is well established. It is therefore useful from the outset to define the following list of functions:

$$Q(x)=\prod_{n=1}^{\infty}(1-x^n) \tag{2.1a}$$

$$P(x)=Q(x)/Q(x^2)=\prod_{n=1}^{\infty}(1-x^{2n-1}) \tag{2.1b}$$

$$R(x)=P(-x)Q(-x)=P^2(-x)Q(x^2)=1+2\sum_{n=1}^{\infty}x^{n^2} \tag{2.1c}$$

$$G(x)=\prod_{n=1}^{\infty}[(1-x^{5n-4})(1-x^{5n-1})]^{-1} \tag{2.1d}$$

$$H(x)=\prod_{n=1}^{\infty}[(1-x^{5n-3})(1-x^{5n-2})]^{-1} \tag{2.1e}$$

$$\begin{aligned}G_1(x)&=[\tfrac{1}{2}(5-\sqrt{5})]^{-1/2}\prod_{n=1}^{\infty}[1+\tfrac{1}{2}(1-\sqrt{5})x^n+x^{2n}]^{-1}\\&=\left(2\sin\left(\frac{\pi}{5}\right)\prod_{n=1}^{\infty}(1-zx^n)(1-z^4x^n)\right)^{-1}\end{aligned} \tag{2.1f}$$

$$\begin{aligned}H_1(x)&=[\tfrac{1}{2}(5+\sqrt{5})]^{-1/2}\prod_{n=1}^{\infty}[1+\tfrac{1}{2}(1+\sqrt{5})x^n+x^{2n}]^{-1}\\&=\left(2\sin\left(\frac{2\pi}{5}\right)\prod_{n=1}^{\infty}(1-z^2x^n)(1-z^3x^n)\right)^{-1}\end{aligned} \tag{2.1g}$$

where $z=\exp(2\pi\mathrm{i}/5)$. These functions are all standard in the literature on Rogers–Ramanujan identities and will occur repeatedly throughout this paper. For later reference we observe that

$$G(x)H(x)=Q(x^5)/Q(x) \tag{2.1h}$$

$$G_1(x)H_1(x)=5^{-1/2}Q(x)/Q(x^5). \tag{2.1i}$$

Using these standard functions, the known results (equations (14.6.7) of Baxter (1982) and (4.31) of Baxter and Pearce (1983)) for the density in the four regimes of the generalised hard hexagon model can be summarised as follows:

$$\rho_{\rm I} = H_1(t^{2/3})H_1(-t)P(-t^{5/3})/P(-t^{15/3}) \tag{2.2a}$$

$$\begin{aligned}\rho_{\rm II} &= \tfrac{1}{3}(\rho_1+\rho_2+\rho_3) = \tfrac{1}{3}(\rho_1-\rho_2)+\rho_2\\ &= -5^{-1/2}t^{1/9}Q(-t)Q(-t^5)/Q(-t^{5/3})\\ &\quad + H_1(-t^{1/9})H_1(-t)Q(-t^{5/9})Q(-t^5)/Q^2(-t^{5/3})\end{aligned} \tag{2.2b}$$

$$\rho_{\rm III}^{\rm fluid} = D(t^{1/4}) = H_1(t^{1/4})H_1(t)/P^2(-t^{5/4}) \tag{2.2c}$$

$$\rho_{\rm III}^{\rm solid} = \tfrac{1}{2}(\rho_1+\rho_2) = D(-t^{1/4}) = H_1(-t^{1/4})H_1(t)/P^2(t^{5/4}) \tag{2.2d}$$

$$\rho_{\rm IV} = \tfrac{1}{2}(\rho_1+\rho_2) = D(t) \equiv H_1(t)H_1(t^4)/P^2(-t^5). \tag{2.2e}$$

Here t is defined by the restriction (1.1) and the relation

$$z^{-1/2}(1-z\,{\rm e}^{L+M}) = \pm[H_1(t)/G_1(t)]^{5/2} \tag{2.3}$$

where the plus sign is taken in regimes I, II and the negative sign in regimes III, IV. The parameter t lies in the range $-1<t<1$ and measures the deviation from the critical lines (1.2) which are the curves corresponding to $t=0$; t is positive in regimes I, III and negative in regimes II, IV. The curves (2.3) of constant t fill the special surface (1.1), the densities (2.2) being constant along these curves.

In all the above cases the critical (multicritical) density, obtained by setting $t=0$, is

$$\rho_{\rm c} = \tfrac{1}{2}(1-5^{-1/2}). \tag{2.4}$$

The results we prove in this paper can now be stated as follows:

$$\rho_{\rm I} = \rho_{\rm c} - (t^{2/3}/\sqrt{5})(G(-t^{1/3})H(t^2)P^2(-t^{5/3})/G(t^{2/3})G(-t)P^2(-t^5)) \tag{2.5a}$$

$$= \rho_{\rm c} - \frac{t^{2/3}}{\sqrt{5}}\left(\frac{H(t^{10/3})Q(-t)P(-t^{5/3})}{G(-t^5)Q(-t^5)P(-t^5)} - t^{1/3}\frac{H(-t^5)}{G(-t^5)}\right) \tag{2.5b}$$

$$\rho_{\rm II} = \rho_{\rm c} + (t^{2/3}/\sqrt{5})(H^2(-t)/G(-t^{1/3})G(-t))(Q^2(-t^5)/Q^2(-t^{5/3})) \tag{2.5c}$$

$$\rho_{\rm I} - \rho_{\rm II} = -(2t^{2/3}/\sqrt{5})(Q(-t)Q(t^{10})/Q(-t^{5/3})Q(t^{10/3})P^2(-t^5)) \tag{2.5d}$$

$$\begin{aligned}\rho_{\rm III}^{\rm fluid} &= D(t^{1/4}) = \tfrac{1}{2}\left(1-\frac{1}{\sqrt{5}}\frac{R(t^{1/4})}{R(t^{5/4})}\right)\\ &= \rho_{\rm c} - (t^{1/4}/\sqrt{5})(H(t^{1/4})H(t)Q(t^{1/2})/R(t^{5/4}))\end{aligned} \tag{2.5e,f}$$

$$\rho_{\rm III}^{\rm solid} = D(-t^{1/4}) = \rho_{\rm c} + (t^{1/4}/\sqrt{5})(H(-t^{1/4})H(t)Q(t^{1/2})/R(-t^{5/4})) \tag{2.5g}$$

$$\rho_{\rm IV} = D(t) = \rho_{\rm c} - (t/\sqrt{5})(H(t)H(t^4)Q(t^2)/R(t^5)) \tag{2.5h}$$

$$\rho_{\rm III}^{\rm fluid} - \rho_{\rm IV} = D(t^{1/4}) - D(t) = -(t^{1/4}/\sqrt{5})(Q(t)Q(t^5)/R(t^{5/4})R(t^5)) \tag{2.5i}$$

$$\rho_{\rm III}^{\rm solid} - \rho_{\rm IV} = D(-t^{1/4}) - D(t) = (t^{1/4}/\sqrt{5})(Q(t)Q(t^5)/R(-t^{5/4})R(t^5)). \tag{2.5j}$$

Equations (2.5*g*) and (2.5*h*) are not independent results; they follow from the functional form of D given in (2.5*f*). Similarly, (2.5*j*) follows from (2.5*i*) by replacing $t^{1/4}$ with $-t^{1/4}$.

We have already said that Huse (1983) has put forward scaling arguments for the behaviour of the singular part $\rho_{\rm sing}$ of the density in the various regimes. Precisely, Huse suggests that $\rho_{\rm sing}$ is of the form

$$\rho_{\rm sing}=t^{2/3}X_{\rm I}(t)\ Y_{\rm I,II}(t^{5/3}) \tag{2.6a}$$

in regimes I, II and

$$\rho_{\rm sing}=t^{1/4}X_{\rm III}(t)Y_{\rm III,IV}(t^{5/4}) \tag{2.6b}$$

in regimes III, IV, where each of the X and Y functions is analytic.

One possible definition of $\rho_{\rm sing}$ is the difference between its value for $t>0$ and the value obtained by analytically continuing from $t<0$ around the singularity at $t=0$. From (2.5d, i, j) we immediately find the simple scaling forms

$$\rho_{\rm I}-\rho_{\rm II}=5^{-1/2}t^{2/3}Q(-t)Y_{\rm I,II}(t^{5/3}) \tag{2.7a}$$

$$\rho_{\rm III}^{\rm fluid}-\rho_{\rm IV}=5^{-1/2}t^{1/4}Q(t)Y_{\rm III,IV}^{\rm fluid}(t^{5/4}) \qquad \rho_{\rm III}^{\rm solid}-\rho_{\rm IV}=5^{-1/2}t^{1/4}Q(t)Y_{\rm III,IV}^{\rm solid}(t^{5/4}) \tag{2.7b, c}$$

where

$$Y_{\rm I,II}(x)=-2Q(x^6)/Q(-x)Q(x^2)P^2(-x^3) \tag{2.7d}$$

$$Y_{\rm III,IV}^{\rm fluid}(x)=-Q(x^4)/R(x)R(x^4) \tag{2.7e}$$

$$Y_{\rm III,IV}^{\rm solid}(x)=Q(x^4)/R(-x)R(x^4)=-Y_{\rm III,IV}^{\rm fluid}(-x) \tag{2.7f}$$

are in fact Taylor expandable functions about $x=0$ with *integer* coefficients. These forms thus confirm (2.6) when $\rho_{\rm sing}$ is defined by analytic continuation.

Although it is far from obvious for $\rho_{\rm II}$, each of the densities in (2.2) can be Taylor expanded about $t=0$ in powers of either $t^{1/3}$ (regimes I, II) or $t^{1/4}$ (regimes III, IV). The singular part of such functions can be conveniently obtained by so expanding and subtracting off all the integer powers of t, that is by using the definition

$$\rho=\rho_{\rm sing}+\rho_{\rm anal} \tag{2.8}$$

where the analytic part $\rho_{\rm anal}$ consists of all the integer powers of t. Suppose $L(x)$ is Taylor expandable and given by

$$L(x)=\sum_{n=0}^{\infty}l_n x^n. \tag{2.9a}$$

Then it will be convenient to define other derived functions by the Taylor expansions

$$[L(x)]_{k({\rm mod}\,p)}=\sum_{n=0}^{\infty}l_{np+k}x^{np+k} \tag{2.9b}$$

$$[L(x)]_{k,p}=L(x)-[L(x)]_{k({\rm mod}\,p)}=\sum_{\substack{j=0\\ j\neq k}}^{p-1}\sum_{n=0}^{\infty}l_{np+j}x^{np+j} \tag{2.9c}$$

where k and p are integers with $0\leqslant k\leqslant p-1$.

Using the alternative definition (2.8) of $\rho_{\rm sing}$ and (2.9) we now find from (2.5b, d, i, j, k) respectively that

$$(\rho_{\rm I})_{\rm sing}=[\rho_{\rm I}(t^{1/3})]_{0,3}=5^{-1/2}t^{2/3}Q(-t)Y_{\rm I}(t^{5/3}) \tag{2.10a}$$

$$(\rho_{\rm II})_{\rm sing}=[\rho_{\rm II}(t^{1/3})]_{0,3}=5^{-1/2}t^{2/3}Q(-t)Y_{\rm II}(t^{5/3}) \tag{2.10b}$$

$$(\rho_{\rm III}^{\rm fluid})_{\rm sing}=[\rho_{\rm III}^{\rm fluid}(t^{1/4})]_{0,4}=5^{-1/2}t^{1/4}Q(t)Y_{\rm III}^{\rm fluid}(t^{5/4}) \tag{2.10c}$$

$$(\rho_{\rm III}^{\rm solid})_{\rm sing}=[\rho_{\rm III}^{\rm solid}(t^{1/4})]_{0,4}=5^{-1/2}t^{1/4}Q(t)Y_{\rm III}^{\rm solid}(t^{5/4}) \tag{2.10d}$$

$$(\rho_{\rm IV})_{\rm sing}=[\rho_{\rm IV}(t^{1/4})]_{0,4}=0 \tag{2.10e}$$

where

$$Y_{\rm I}(x)=-[H(x^2)P(-x)/G(-x^3)Q(-x^3)P(-x^3)]_{1,3} \tag{2.10f}$$

$$Y_{\rm II}(x)=Y_{\rm I}(x)+2[Q(x^6)/Q(-x)Q(x^2)P^2(-x^3)]_{1,3} \tag{2.10g}$$

$$Y_{\rm III}^{\rm fluid}(x)=-[Q(x^4)/R(x)R(x^4)]_{3,4} \tag{2.10h}$$

$$Y_{\rm III}^{\rm solid}(x)=[Q(x^4)/R(-x)R(x^4)]_{3,4}=-Y_{\rm III}^{\rm fluid}(-x) \tag{2.10i}$$

are once again all Taylor expandable functions with *integer* coefficients. Clearly, the results (2.7) and (2.10) firmly establish the simple scaling forms (2.6) conjectured by Huse (1983) with

$$X_{\rm I}(t)=5^{-1/2}Q(-t) \qquad X_{\rm III}(t)=5^{-1/2}Q(t). \tag{2.11a, b}$$

3. The identities

It remains to prove the product forms (2.5). To do this it suffices to prove a series of six identities as follows:

$$H_1(x^2)H_1(x^3)\frac{P(x^5)}{P(x^{15})}=\frac{1}{2}-\frac{1}{\sqrt{5}}\left(\frac{1}{2}+x^2\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P^2(x^5)}{P^2(x^{15})}\right) \tag{3.1a}$$

$$\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P^2(x^5)}{P^2(x^{15})}=\frac{H(x^{10})}{G(x^{15})}\frac{Q(x^3)}{Q(x^{15})}\frac{P(x^5)}{P(x^{15})}+x\frac{H(x^{15})}{G(x^{15})} \tag{3.1b}$$

$$H_1(x^{1/3})H_1(x^3)\frac{Q(x^{5/3})Q(x^{15})}{Q(x^5)}$$
$$=\frac{1}{2}-\frac{1}{\sqrt{5}}\left(\frac{1}{2}+x^{1/3}\frac{Q(x^3)Q(x^{15})}{Q^2(x^5)}-x^2\frac{H^2(x^3)}{G(x)G(x^3)}\frac{Q^2(x^{15})}{Q^2(x^5)}\right) \tag{3.1c}$$

$$\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P^2(x^5)}{P^2(x^{15})}+\frac{H^2(x^3)}{G(x)G(x^3)}\frac{Q^2(x^{15})}{Q^2(x^5)}=2\frac{Q(x^3)Q(x^{30})}{Q(x^5)Q(x^{10})P^2(x^{15})} \tag{3.1d}$$

$$D(x)\equiv\frac{H_1(x)H_1(x^4)}{P^2(-x^5)}=\frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\frac{R(x)}{R(x^5)}\right)=\frac{1}{2}-\frac{1}{\sqrt{5}}\left(\frac{1}{2}+xH(x)H(x^4)\frac{Q(x^2)}{R(x^5)}\right) \tag{3.1e}$$

$$D(x)-D(x^4)=\frac{1}{2\sqrt{5}}\left(\frac{R(x^4)}{R(x^{20})}-\frac{R(x)}{R(x^3)}\right)=-\frac{x}{\sqrt{5}}\frac{Q(x^4)Q(x^{20})}{R(x^5)R(x^{20})}. \tag{3.1f}$$

The first four identities, with $x=-t^{1/3}$, pertain to the regime I, II results (2.5*a–d*). The last two compound identities, with $x=\pm t^{1/4}$ or t, are all that is needed to establish the regime III, IV results (2.5*e–j*).

The six identities (3.1) were first obtained from computer calculations. In the case of (3.1*a, c, e*), a Fortran program was used to evaluate the first 95 or so coefficients in a Taylor expansion of the densities (2.2) in the form

$$\rho(x)/\rho_c = 1 + \sum_{n=1}^{\infty} (a_n^1 z + a_n^2 z^2 + a_n^3 z^3 + a_n^4 z^4)x^n \tag{3.2a}$$

where $z = \exp(2\pi i/5)$, $x = t^{1/3}$, $\pm t^{1/4}$ or t as appropriate and a_n^1, a_n^2, a_n^3, a_n^4 with $n = 1, 2, \ldots, 95$ are stored integers. All one needs is subroutines for multiplying and dividing by $(1 \pm z^m x^n)$. Sure enough, for all the coefficients evaluated, it was found that

$$a_n^1 = a_n^4 = a_n^2 - a_n^3 = 0 \tag{3.2b}$$

which strongly suggests that $\rho(x)$ is of the form

$$\rho(x) = \tfrac{1}{2}(1 - 5^{-1/2}) - 5^{-1/2} \sum_{n=1}^{\infty} b_n x^n \tag{3.2c}$$

where the b_n are all integers and $b_n = a_n^2$ for $n = 1, 2, \ldots, 100$. Next, a subroutine was used to convert the series into product form

$$\sum_{n=1}^{\infty} b_n x^n = x^{c_0} \prod_{n=1}^{\infty} (1 - x^n)^{c_n} \tag{3.2d}$$

evaluating the first 95 integer exponents $c_0, c_1, \ldots, c_{94}$. In every case, obvious recurrence patterns in these exponents, repeated over many periods, enabled us to identify the product form (3.2*d*) as well defined products of the elliptic functions (2.1*a*–*e*).

The identities (3.1*d*, *f*) were similarly obtained by using (3.2*c*) to evaluate $\rho_{\rm I} - \rho_{\rm II}$ and $\rho_{\rm III} - \rho_{\rm IV}$ (fluid or solid) in series form and then using (3.2*d*) to convert to the product form. The remaining identity (3.1*b*) was more difficult to obtain. First we obtained the series for the singular part of $\rho_{\rm I}$ numerically, converted to product form and found, in agreement with (2.6*a*), that to order x^{95}

$$[\rho_{\rm I}(x)]_{0,3} = -5^{-1/2} x^2 Q(x^3) A(x^5) \tag{3.3a}$$

where the function A has a Taylor expansion with integer coefficients but no simple product form. Using (2.5*a*) this leads to the functional decomposition

$$(G(x)H(x^6)/G(x^2)G(x^3))(P^2(x^5)/P^2(x^{15})) = Q(x^3)A(x^5) + xB(x^3) \tag{3.3b}$$

where B is also a Taylor expandable function with integer coefficients. The decomposition (3.3*b*), however, is not unique. Indeed, by redistributing the terms in the series for the right-hand side, it was found possible, this time working to order x^{190}, to arrange that

$$(G(x)H(x^6)/G(x^2)G(x^3))(P^2(x^5)/P^2(x^{15})) = Q(x^3)Y(x^5) + xC(x^{15}) \tag{3.3c}$$

where now the Taylor expandable functions Y and C are determined *uniquely* and, moreover, are found numerically to have a simple product form leading to the identity (3.1*b*). Although the singular part of $\rho_{\rm II}$ is also of the form (3.3*a*), we found no simplifying feature analogous to (3.3*c*).

Having used machine calculations to guess the form of the six identities (3.1), it remains to prove them analytically. This we do in the remainder of this section, proving the six identities in the order listed in (3.1).

3.1. Proof of first identity

To prove the identity (3.1a) we will actually prove the pair of identities

$$H_1(x^2)H_1(x^3)\frac{P(x^5)}{P(x^{15})}=\frac{1}{2}-\frac{1}{\sqrt{5}}\left(\frac{1}{2}+x^2\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P^2(x^5)}{P^2(x^{15})}\right) \tag{3.4a}$$

$$G_1(x^2)G_1(x^3)\frac{P(x^5)}{P(x^{15})}=\frac{1}{2}+\frac{1}{\sqrt{5}}\left(\frac{1}{2}+x^2\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P^2(x^5)}{P^2(x^{15})}\right). \tag{3.4b}$$

From (2.1) these identities are equivalent under the interchange of $\sqrt{5}$ and $-\sqrt{5}$. Adding and subtracting the identities (3.4) we obtain

$$G_1(x^2)G_1(x^3)+H_1(x^2)H_1(x^3)=P(x^{15})/P(x^5) \tag{3.5a}$$

$$\sqrt{5}[G_1(x^2)G_1(x^3)-H_1(x^2)H_1(x^3)]=\frac{P(x^{15})}{P(x^5)}+2x^2\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P(x^5)}{P(x^{15})}. \tag{3.5b}$$

Using (3.5a), this last identity can be written as

$$\tfrac{1}{2}(\sqrt{5}-1)G_1(x^2)G_1(x^3)-\tfrac{1}{2}(\sqrt{5}+1)H_1(x^2)H_1(x^3)=x^2\frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P(x^5)}{P(x^{15})}. \tag{3.5c}$$

At this stage we need to introduce some more elliptic functions in order to convert (3.5) into series form. For $|q|<1$, $0<|w|<\infty$ we therefore define

$$f(w,q)=\prod_{n=1}^{\infty}(1-q^{n-1}w)(1-q^nw^{-1})(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n-1)/2}w^n=f(qw^{-1},q) \tag{3.6}$$

$$\theta_1(u,q)=2\sin u\prod_{n=1}^{\infty}(1-2q^n\cos 2u+q^{2n})(1-q^n)=-\mathrm{i}\sum_{n=-\infty}^{\infty}(-1)^nq^{n(n+1)/2}\,\mathrm{e}^{(2n+1)u\mathrm{i}}. \tag{3.7}$$

These functions are related by

$$\theta_1(u,q)=\mathrm{i}\,\mathrm{e}^{-\mathrm{i}u}f(\mathrm{e}^{2\mathrm{i}u},q). \tag{3.8}$$

The definition (3.7) of the elliptic theta function is non-standard and differs from Baxter (1982) by a factor of 2. From (2.1) and (3.7) we see immediately that

$$\sqrt{5}Q(x^5)G_1(x)=Q(x)/H_1(x)=\theta_1(2\pi/5,x) \tag{3.9a}$$

$$\sqrt{5}Q(x^5)H_1(x)=Q(x)/G_1(x)=\theta_1(4\pi/5,x). \tag{3.9b}$$

Using (3.9), the identities (3.5a, c) can now be written as

$$5^{-1}\sum_{p=-2}^{2}\theta_1(2\pi p/5,x^2)\theta_1(2\pi p/5,x^3)=f(-1,x^5)f(x^{15},x^{30}) \tag{3.10a}$$

$$5^{-1}\sum_{p=-2}^{2}z^{-4p}\theta_1(2\pi p/5,x^2)\theta_1(2\pi p/5,x^3)=x^2f(-x,x^5)f(x^3,x^{30}) \tag{3.10b}$$

where again $z=\exp(2\pi\mathrm{i}/5)$ and we have used the simple facts that

$$z+z^4=2\cos(2\pi/5)=\tfrac{1}{2}(\sqrt{5}-1) \qquad z^2+z^3=2\cos(4\pi/5)=-\tfrac{1}{2}(\sqrt{5}+1) \tag{3.11a, b}$$

$$\theta_1(0,q)=0 \qquad \theta_1(-u,q)=-\theta_1(u,q) \tag{3.12}$$

as well as the straightforward identities

$$2P(x^3)/P(x)=f(-1,x)f(x^3,x^6)/Q(x^2)Q(x^3) \tag{3.13}$$

$$(G(x)H(x^6)/G(x^2)G(x^3))\,(P(x^5)/P(x^{15}))=f(-x,x^5)f(x^3,x^{30})/Q(x^{10})Q(x^{15}). \tag{3.14}$$

The identities (3.10) can now be proved by manipulating the double series representations as follows:

$$\begin{aligned}
5^{-1}\sum_{p=-2}^{2} & z^{2np}\theta_1(2\pi p/5,x^2)\theta_1(2\pi p/5,x^3)\\
&=5^{-1}\sum_{p=-2}^{2}\sum_{r,s=-\infty}^{\infty}(-1)^{r+s+1}x^{r(r+1)+3s(s+1)/2}z^{2p(r+s+n+1)}\\
&=\sum_{k,s=-\infty}^{\infty}(-1)^{k+n}x^{(5k-s-n-\frac{1}{2})^2+3(s+\frac{1}{2})^2/2-\frac{5}{8}}\\
&=\sum_{k,t=-\infty}^{\infty}(-1)^{k+n}x^{5t(t-1)/2+(2n+5)t+15k(k-1)+(15-6n)k+n(n+1)}\\
&=(-1)^n x^{n(n+1)}f(-x^{2n+5},x^5)f(x^{15-6n},x^{30}).
\end{aligned} \tag{3.15}$$

Here we have used the series representations (3.7) of the theta functions and restricted the sum on r to the values $r=5k-s-n-1$ because of the relation

$$5^{-1}\sum_{p=-2}^{2} z^{mp}=\begin{cases}1, & m=0\ (\mathrm{mod}\ 5),\\ 0, & m\neq 0\ (\mathrm{mod}\ 5).\end{cases} \tag{3.16}$$

Next, we factored the double series by transforming from s to $t=s-2k$. The last step in (3.15) then follows from (3.6). Finally, setting $n=0,-2$ in (3.15) yields (3.10*a*, *b*) respectively.

Before proceeding, it should be pointed out that the identity (3.5*a*) is actually the conjugate modulus form (see the appendix) of the standard identity

$$G(x^2)G(x^3)+xH(x^2)H(x^3)=P(x^3)/P(x) \tag{3.17}$$

which is identity (7) on the list of Ramanujan's 40 'sums of products' identities given by Birch (1975). From the conjugate modulus form of (3.5*b*) we see that we have proved the complementary identity

$$\tfrac{1}{2}(\sqrt{5}-1)G(x^2)G(x^3)-\tfrac{1}{2}(\sqrt{5}+1)xH(x^2)H(x^3)=\frac{H_1(x)G_1(x^6)}{G_1(x^2)G_1(x^3)}\frac{P(x)}{P(x^3)}. \tag{3.18}$$

3.2. *Proof of second identity*

After rearranging, the identity (3.1*b*) can be written as

$$\begin{aligned}
Q(x^{10})Q(x^{15}) & \frac{G(x)H(x^6)}{G(x^2)G(x^3)}\frac{P(x^5)}{P(x^{15})}\frac{Q(x^{75})}{H(x^{15})}\\
&=Q(x^3)Q(x^{15})Q(x^{10})H(x^{10})\\
&\quad +xQ(x^{10})Q(x^{15})P(x^{15})Q(x^{75})/P(x^5)G(x^{15}).
\end{aligned} \tag{3.19}$$

Using (3.14) and the identities

$$Q(x)G(x)=Q(x^5)/H(x)=f(x^2,x^5) \qquad Q(x)H(x)=Q(x^5)/G(x)=f(x,x^5) \tag{3.20a, b}$$

$$Q(x)=f(x,x^3) \tag{3.21}$$

$$Q(x^3)P(x^3)/P(x)=f(-x,x^3) \tag{3.22}$$

this becomes

$$f(-x,x^5)f(x^3,x^{30})f(x^{30},x^{75})=f(x^3,x^{15})f(x^6,x^{15})f(x^{10},x^{50}) \\ +xf(x^{10},x^{30})f(-x^5,x^{15})f(x^{15},x^{75}). \tag{3.23}$$

To prove the identity (3.23) we first prove the two auxiliary identities

$$f(-x,x^5)f(x^3,x^{30})=f(x^{10},x^{50})f(x^{30},x^{75})+xf(x^{20},x^{50})f(x^{15},x^{75}) \\ -x^3f(x^{10},x^{50})f(x^{15},x^{75}) \tag{3.24}$$

$$f(x,x^5)f(x^2,x^5)=f^2(x^{10},x^{25})-xf(x^5,x^{25})f(x^{10},x^{25})-x^2f^2(x^5,x^{25}). \tag{3.25}$$

The double series for the left-hand side (LHS) of (3.24) is

$$\sum_{r,s=-\infty}^{\infty}(-1)^s x^{5r(r-1)/2+r+15s(s-1)+3s}$$

$$=\sum_{p=-2}^{2}\sum_{k,s=-\infty}^{\infty}(-1)^s x^{5(2s+5k+p)^2/2-\frac{3}{2}(2s+5k+p)+15s^2-12s}$$

$$=\sum_{p=-2}^{2}\sum_{k,t=-\infty}^{\infty}(-1)^{k+t}x^{25t(t-1)+10(p+1)t+75k(k-1)/2+15(p+3)k+(5p^2-3p)/2}$$

$$=\sum_{p=-2}^{2}x^{(5p^2-3p)/2}f(x^{10p+10},x^{50})f(x^{15p+45},x^{75}). \tag{3.26}$$

In the first step we have held s fixed and summed separately over the values $r=2s+5k+p$ for $p=-2,\ -1,\ 0,1,2$. Next we transformed from s to $t=s+k$ and lastly we factored the double series using (3.6). Writing out the five terms in the last sum and using the simple results

$$f(1,q)=0 \qquad f(w^{-1},q)=f(qw,q)=-w^{-1}f(w,q) \tag{3.27}$$

we obtain the RHS of (3.24). Similarly, the double series for the LHS of (3.25) is

$$\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}x^{5r(r-1)/2+r+5s(s-1)/2+2s}$$

$$=\sum_{p=-2}^{2}\sum_{k,s=-\infty}^{\infty}(-1)^{k+s+p}x^{5(5k-2s+p)^2/2-3(5k-2s+p)/2+5s^2/2-3s/2}$$

$$=\sum_{p=-2}^{2}\sum_{k,t=-\infty}^{\infty}(-1)^{k+t+p}x^{25t(t-1)/2+5(3-2p)t+25k(k-1)/2+5(2+p)k+(5p^2-3p)/2}$$

$$=\sum_{p=-2}^{2}(-1)^p x^{(5p^2-3p)/2}f(x^{15-10p},x^{25})f(x^{10+5p},x^{25}) \tag{3.28}$$

where this time $t=s-2k$. Writing out this last sum using (3.27) gives the RHS of (3.25).

We now return to the identity (3.23). Putting (3.24) and (3.25), with x replaced by x^3, into (3.23) we see that both sides of this equation are of the form

$$A(x^5)+xB(x^5)+x^3C(x^5). \tag{3.29}$$

Equating the corresponding functions of x^5 on each side, and replacing x^5 with x, we are left with the single identity

$$f(x^4,x^{10})f(x^6,x^{15})=f(-x,x^3)f(x^2,x^6)-xf(x^2,x^{10})f(x^3,x^{15}) \tag{3.30}$$

the other two being satisfied trivially. But using (3.20), (3.21) and (3.22) this last identity is just the Ramanujan identity (3.17) stated by Birch (1975).

3.3. *Proof of third identity*

Replacing x in (3.1c) with x^3 and then $\sqrt{5}$ with $-\sqrt{5}$ we obtain the pair of identities

$$H_1(x)H_1(x^9)\frac{Q(x^5)Q(x^{45})}{Q^2(x^{15})}=\frac{1}{2}-\frac{1}{\sqrt{5}}\left(\frac{1}{2}+x\frac{Q(x^9)Q(x^{45})}{Q^2(x^{15})}-x^6\frac{H^2(x^9)}{G(x^3)G(x^9)}\frac{Q^2(x^{45})}{Q^2(x^{15})}\right) \tag{3.31a}$$

$$G_1(x)G_1(x^9)\frac{Q(x^5)Q(x^{45})}{Q^2(x^{15})}=\frac{1}{2}+\frac{1}{\sqrt{5}}\left(\frac{1}{2}+x\frac{Q(x^9)Q(x^{45})}{Q^2(x^{15})}-x^6\frac{H^2(x^9)}{G(x^3)G(x^9)}\frac{Q^2(x^{45})}{Q^2(x^{15})}\right). \tag{3.31b}$$

Adding and subtracting these we find

$$G_1(x)G_1(x^9)+H_1(x)H_1(x^9)=Q^2(x^{15})/Q(x^5)Q(x^{45}) \tag{3.32a}$$

$$\sqrt{5}[G_1(x)G_1(x^9)-H_1(x)H_1(x^9)]=\frac{Q^2(x^{15})}{Q(x^5)Q(x^{45})}+2x\frac{Q(x^9)}{Q(x^5)}-2x^6\frac{H^2(x^9)}{G(x^3)G(x^9)}\frac{Q(x^{45})}{Q(x^{15})}. \tag{3.32b}$$

Using (3.32a), this last identity becomes

$$\tfrac{1}{2}(\sqrt{5}-1)G_1(x)G_1(x^9)-\tfrac{1}{2}(\sqrt{5}+1)H_1(x)H_1(x^9)=x\frac{Q(x^9)}{Q(x^5)}-x^6\frac{Q(x^{45})}{Q(x^5)}\frac{H^2(x^9)}{G(x^3)G(x^9)}. \tag{3.32c}$$

Using (3.9), (3.11) and (3.12) the identities (3.3a, c) can be written as

$$5^{-1}\sum_{p=-2}^{2}\theta_1(2\pi p/5,x)\theta_1(2\pi p/5,x^9)=2Q^2(x^{15}) \tag{3.33a}$$

$$5^{-1}\sum_{p=-2}^{2}z^{-4p}\theta_1(2\pi p/5,x)\theta_1(2\pi p/5,x^9)$$
$$=xQ(x^9)Q(x^{45})-x^6Q^2(x^{45})H^2(x^9)/G(x^3)G(x^9). \tag{3.33b}$$

To prove these we now manipulate the double series representations as follows:

$$5^{-1}\sum_{p=-2}^{2}z^{2np}\theta_1(2\pi p/5,x)\theta_1(2\pi p/5,x^9)$$
$$=-5^{-1}\sum_{p=-2}^{2}\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}x^{r(r+1)/2+9s(s+1)/2}z^{2p(r+s+n+1)}$$

$$= \sum_{k,s=-\infty}^{\infty} (-1)^{k-n} x^{[5(5k^2-2ks+2s^2)-5(2n+1)k+2(n+5)s+n(n+1)]/2}$$

$$= \sum_{l=-1}^{1} \sum_{m,t=-\infty}^{\infty} (-1)^{m+t+5l+n} x^{[45t(t-1)+12(5l+n+5)t]/2}$$

$$\times x^{[45m(m-1)+6(10-5l-n)m+(5l+n)(5l+n+1)]/2}$$

$$= \sum_{l=-1}^{1} (-1)^{5l+n} x^{(5l+n)(5l+n+1)/2} f(x^{30l+6n+30}, x^{45}) f(x^{30-15l-3n}, x^{45}). \qquad (3.34)$$

Here we have used the definition (3.7) and restricted the sum on r to the values $r=5k-s-n-1$ using (3.16). Next we have set $k=3m-s-l=m-t-l$ with $t=s-2m$ and summed separately over $l=-1, 0, 1$. Finally, for each l, the double series factors into a product of f-functions given by (3.6).

Setting $n=0$ in (3.34) and using (3.27) and (3.21) we obtain (3.33a). This proves (3.32a) which is the conjugate modulus form (appendix) of the Ramanujan identity

$$G(x)G(x^9)+x^2H(x)H(x^9)=Q^2(x^3)/Q(x)Q(x^9) \qquad (3.35)$$

which is identity (6) on the list given by Birch (1975). Setting $n=-2$, $l=0$ in (3.34) we obtain, using (3.20), the term

$$xf(x^{18}, x^{45})f(x^9, x^{45})=xQ(x^9)Q(x^{45}) \qquad (3.36)$$

as in (3.33b). Evaluating the remaining terms ($n=-2$, $l=-1$, 1) in (3.34), using (3.27), and comparing with (3.33b), we find that it remains to show

$$f(x^2, x^{15})f(x^4, x^{15})-f(x, x^{15})f(x^8, x^{15})=xQ^2(x^{15})H^2(x^3)/G(x)G(x^3) \qquad (3.37)$$

where we have replaced x^3 with x. But using the simple identity

$$Q^2(x^{15})H^2(x^3)/G(x)G(x^3)=f(x, x^{15})f(x^4, x^{15})f(x^3, x^{15})/f(x^6, x^{15}) \qquad (3.38)$$

we see that (3.37) is just a special case ($w=x$, $q=x^5$) of the general identity

$$f(w^2, q^3)f(qw, q^3)f(qw^{-1}, q^3)-wf(w, q^3)f(q^2w^2, q^3)f(qw^{-1}, q^3)$$
$$=f(w, q^3)f(qw, q^3)f(q^2w^{-2}, q^3). \qquad (3.39)$$

To prove (3.39) we fix q and let $F(w)$, with w complex, be the ratio of the LHS over the RHS. We now observe that $F(w)=F(q^3w)$ is analytic throughout a period annulus and hence is constant by Liouville's theorem. Setting $w=-1$ verifies that the constant is unity. This completes the proof of (3.33) and hence establishes (3.31).

3.4. *Proof of fourth identity*

To prove the identity (3.1d) we begin by observing the following straightforward identities:

$$(H(x^3)/G(x))(Q^2(x^{15})/Q^2(x^5))=f(x, x^{15})f(x^{11}, x^{15})/f^2(x^5, x^{15}) \qquad (3.40a)$$

$$(G(x)H(x^6)/G(x^2)H(x^3))(P^2(x^5)/P^2(x^{15}))$$
$$=f(-x, x^{15})f(-x^{11}, x^{15})/f^2(x^5, x^{15}) \qquad (3.40b)$$

$$H(x^3)/G(x^3)=f(x^3, x^{15})/f(x^6, x^{15}) \qquad (3.40c)$$

$$2\frac{Q(x^3)Q(x^{30})}{Q(x^5)Q(x^{10})P^2(x^{15})}=\frac{f(-1,x^{15})f(x^3,x^{15})f(x^6,x^{15})}{f(-x^5,x^{15})f^2(x^5,x^{15})}. \tag{3.40d}$$

Putting these in (3.1d) we obtain the identity

$$f^2(-x^5,x^{15})f(x,x^{15})f(x^{11},x^{15})+f^2(x^5,x^{15})f(-x,x^{15})f(-x^{11},x^{15}) \\ =f(-1,x^{15})f(-x^{10},x^{15})f^2(x^6,x^{15}). \tag{3.41}$$

But this is just a special case ($a=b=-x^5$, $c=x$, $q=x^{15}$) of the general identity

$$f(a)f(b)f(c)f(abc)+f(-a)f(-b)f(-c)f(-abc)=f(-1)f(-ab)f(-bc)f(-ca) \tag{3.42}$$

where a, b, c are complex and $f(w)\equiv f(w,q)$.

The identity (3.42) can be proved analogously to (3.39). Let $F(a)$ be the ratio of the LHS of (3.42) over the RHS. Then $F(a)=F(qa)$ is analytic throughout a period annulus and hence is constant by Liouville's theorem. Setting $a=1$ verifies that the constant is unity. This proves (3.41) and hence (3.1d).

3.5. *Proof of fifth identity*

The second and third identities on the list of 40 Ramanujan identities given by Birch (1975) are

$$G(x)G(x^4)+xH(x)H(x^4)=P^2(-x)=R(x)/Q(x^2) \tag{3.43a}$$

$$G(x)G(x^4)-xH(x)H(x^4)=R(x^5)/Q(x^2). \tag{3.43b}$$

In conjugate modulus form (appendix) these become

$$G_1(x)G_1(x^4)+H_1(x)H_1(x^4)=P^2(-x^5)=R(x^5)/Q(x^{10}) \tag{3.44a}$$

$$G_1(x)G_1(x^4)-H_1(x)H_1(x^4)=5^{-1/2}R(x)/Q(x^{10}). \tag{3.44b}$$

From (3.43) we find that

$$xH(x)H(x^4)=\tfrac{1}{2}[R(x)-R(x^5)]/Q(x^2). \tag{3.45}$$

Similarly, from (3.44) we find that

$$H_1(x)H_1(x^4)=\tfrac{1}{2}[R(x^5)-5^{-1/2}R(x)]/Q(x^{10}). \tag{3.46}$$

It follows that

$$D(x)=\frac{H_1(x)H_1(x^4)}{P^2(-x^5)}=\frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\frac{R(x)}{R(x^5)}\right)=\frac{1}{2}-\frac{1}{\sqrt{5}}\left(\frac{1}{2}+xH(x)H(x^4)\frac{Q(x^2)}{R(x^5)}\right) \tag{3.47}$$

which is (3.1e).

3.6. *Proof of sixth identity*

From (3.47) we see immediately that

$$D(x)-D(x^4)=(1/2\sqrt{5})[R(x^4)/R(x^{20})-R(x)/R(x^5)] \tag{3.48}$$

which is the first part of (3.1f). Also, from the two identities (3.43), one can establish that

$$R(x)R(x^{20})-R(x^4)R(x^5)=2xQ(x^2)Q(x^8)G(x^4)H(x^4)[G(x^{16})H(x)-x^3G(x)H(x^{16})]. \tag{3.49}$$

But now identity (5) on the list of Birch (1975) is

$$G(x^{16})H(x)-x^3G(x)H(x^{16})=P(-x^2)=Q^2(x^4)/Q(x^2)Q(x^8). \tag{3.50}$$

Putting this in (3.49) and using (2.1h) we obtain

$$R(x)R(x^{20})-R(x^4)R(x^5)=2xQ(x^4)Q(x^{20}). \tag{3.51}$$

Finally, combining (3.51) and (3.48), we see that

$$D(x)-D(x^4)=-(x/\sqrt{5})\,(Q(x^4)Q(x^{20})/R(x^5)R(x^{20})). \tag{3.52}$$

This is the second part of (3.1f) and completes the proof of the six identities (3.1).

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Appendix. Conjugate modulus transformations

All the identities in this paper can be written in two equivalent forms. These forms are related by the following conjugate modulus transformations:

$$Q(e^{-\varepsilon})=(2\pi/\varepsilon)^{1/2}\exp(\tfrac{1}{24}\varepsilon-\pi^2/6\varepsilon)Q(e^{-4\pi^2/\varepsilon}) \tag{A1}$$

$$P(e^{-\varepsilon})=\sqrt{2}\exp(-\tfrac{1}{24}\varepsilon-\pi^2/12\varepsilon)/P(e^{-2\pi^2/\varepsilon}) \tag{A2}$$

$$R(e^{-\varepsilon})=(\pi/\varepsilon)^{1/2}R(e^{-\pi^2/\varepsilon}) \tag{A3}$$

$$G(e^{-\varepsilon})=\exp(-\tfrac{1}{60}\varepsilon+\pi^2/15\varepsilon)G_1(e^{-4\pi^2/5\varepsilon}) \tag{A4}$$

$$H(e^{-\varepsilon})=\exp(\tfrac{11}{60}\varepsilon+\pi^2/15\varepsilon)H_1(e^{-4\pi^2/5\varepsilon}) \tag{A5}$$

$$Q(-e^{-\varepsilon})=(\pi/\varepsilon)^{1/2}\exp(\tfrac{1}{24}\varepsilon-\pi^2/24\varepsilon)Q(-e^{-\pi^2/\varepsilon}) \tag{A6}$$

$$P(-e^{-\varepsilon})=\exp(-\tfrac{1}{24}\varepsilon+\pi^2/24\varepsilon)P(-e^{-\pi^2/\varepsilon}) \tag{A7}$$

$$G(-e^{-\varepsilon})=\exp(-\tfrac{1}{60}\varepsilon+\pi^2/60\varepsilon)H_1(-e^{-\pi^2/5\varepsilon}) \tag{A8}$$

$$H(-e^{-\varepsilon})=\exp(\tfrac{11}{60}\varepsilon+\pi^2/60\varepsilon)G_1(-e^{-\pi^2/5\varepsilon}) \tag{A9}$$

$$\theta_1(u,e^{-\varepsilon})=(2\pi/\varepsilon)^{1/2}\exp[\tfrac{1}{8}\varepsilon-\pi^2/2\varepsilon+2u(\pi-u)/\varepsilon]f(e^{-4\pi u/\varepsilon},e^{-4\pi^2/\varepsilon}) \tag{A10}$$

$$\theta_4(u,e^{-\varepsilon})=(2\pi/\varepsilon)^{1/2}\exp[-\pi^2/2\varepsilon+2u(\pi-u)/\varepsilon]f(-e^{-4\pi u/\varepsilon},e^{-4\pi^2/\varepsilon}) \tag{A11}$$

where the last elliptic theta function is defined by

$$\theta_4(u,q^2)=\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2u+q^{4n-2})(1-q^{2n}). \tag{A12}$$

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